q-Identities related to overpartitions and divisor functions

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Abstract. We prove some q-Identities related to overpartitions and divisor functions

1. Introduction

In this note, using the classical notations $(z;q)_i = (1-z)\cdots(1-zq^{i-1})$, and $\begin{bmatrix} n \\ i \end{bmatrix} = (q;q)_n/(q;q)_i(q;q)_{n-i}$, we prove the following two identities :

Theorem 1.1 *For any pair positive integers m, n, one has*

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1}(x+1)\dots(x+q^{i-1})}{(1-q^i)^m} q^{mi} = \sum_{i=1}^{n} \frac{(-1)^{i-1}(x^i-(-1)^i)}{1-q^i} q^i \sum_{i\le i_2\le\ldots\le i_m\le n} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1-q^{i_j})}, \quad (1.1)$$

$$\frac{(z;q)_{n+1}}{(q;q)_n} \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1}(x+1)\dots(x+q^{i-1})}{1-zq^i} q^i = \sum_{i=0}^{n} (-1)^{i-1} \frac{(z;q)_i}{(q;q)_i} x^i q^i \,. \quad (1.2)$$

In the next section, we shall show that (1.1) and (1.2) can be obtained from the Newton interpolation in points $\{-1, -q, -q^2, \ldots\}$, using complete functions in the variables $\{q/(1-q), q^2/(1-q^2), \cdots\}$.

Using the same methods, we have already given in [5] an identity which generalizes the case $x = 0$ of (1.1).

Given $\mathbb{X} = \{x_1, x_2, \ldots\}$, Newton gave the following interpolation formula, for any function $f(x)$:

$$f(x) = f(x_1) + f\partial_1(x - x_1) + f\partial_1\partial_2(x - x_1)(x - x_2) + \ldots,$$

where ∂_i, acting on its left, is defined by

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots)\partial_i = \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}.$$

Taking in particular $f(x) = x^n$, we have

$$x_1^n \partial_1 \ldots \partial_i = h_{n-i}(x_1, x_2, \ldots, x_{i+1}), \tag{1.3}$$

where h_k is the complete function of degree k, defined by

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_k \le n} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

Recall the following properties of h_k:

1. The generating function of h_k is:

$$\sum_{k=0}^{\infty} h_k(x_1, \ldots, x_n) t^k = \frac{1}{(1 - tx_1) \ldots (1 - tx_n)}. \tag{1.4}$$

2. More generally, given two alphabets $\mathbb{X}$ and $\mathbb{Y}$, then the generating functions of $h_k(\mathbb{X} + \mathbb{Y})$ and $h_k(\mathbb{X} - \mathbb{Y})$ are:

$$\begin{aligned} \sum_{k=0}^{\infty} h_k(\mathbb{X} + \mathbb{Y}) t^k &= \frac{1}{\prod_{x \in \mathbb{X}}(1 - xt) \prod_{y \in \mathbb{Y}}(1 - yt)}, \\ \sum_{k=0}^{\infty} h_k(\mathbb{X} - \mathbb{Y}) t^k &= \frac{\prod_{y \in \mathbb{Y}}(1 - yt)}{\prod_{x \in \mathbb{X}}(1 - xt)}. \end{aligned} \tag{1.5}$$

   As a consequence, one has:

$$h_n(\mathbb{X} + \mathbb{Y}) = \sum_{k=0}^{n} h_k(\mathbb{X}) h_{n-k}(\mathbb{Y}). \tag{1.6}$$

3. Given $\{x_1, x_2, \ldots, x_n\}$, and a positive integer m, we have:

$$\sum_{i=1}^{n} x_i h_{m-1}(x_i, x_{i+1}, \ldots, x_n) = h_m(x_1, x_2, \ldots, x_n). \tag{1.7}$$

Taking $\mathbb{X} = \{-1, -q, -q^2, \ldots\}$, it is easy to check from (1.3) and (1.4):

$$x_1^n \partial_1 \ldots \partial_i = h_{n-i}(-1, -q, \ldots, -q^i) = (-1)^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}.$$

2. Proofs of (1.1) and (1.2)

The Gauss polynomials $\begin{bmatrix} n \\ k \end{bmatrix}$ satisfy the following recursion (cf. [1]):

$$\sum_{j=0}^{n} \begin{bmatrix} m+j \\ m \end{bmatrix} q^j = \begin{bmatrix} n+m+1 \\ m+1 \end{bmatrix}. \tag{2.1}$$

In this paper, we need the following more general relations.

Lemma 2.1 *Let k, m and n be nonnegative integers. Then we have the following formulas:*

$$\sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \sum_{i \le i_2 \dots \le i_m \le n} \frac{q^{\sum_{j=2}^{m} i_j}}{\prod_{j=2}^{m}(1-q^{i_j})} = \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{km}}{(1-q^k)^m}, \tag{2.2}$$

and

$$\sum_{i=0}^{n} \frac{(z;q)_i}{(q;q)_i} q^i = \frac{(zq;q)_n}{(q;q)_n}. \tag{2.3}$$

Proof. Taking $\mathbb{X} = \{1, q, \dots, q^l\}$ and $\mathbb{Y} = \{q^{l+1}, q^{l+2}, \dots\}$, we obtain from (1.5) and (1.6):

$$\frac{1}{(q;q)_n} = h_n(\mathbb{X}+\mathbb{Y}) = \sum_{i=0}^{n} h_i(\mathbb{Y}) h_{n-i}(\mathbb{X}) = \sum_{i=0}^{n} \frac{1}{(q;q)_i} \begin{bmatrix} n-i+l \\ l \end{bmatrix} q^{(l+1)i}. \tag{2.4}$$

Letting $f(m)$ be the left side of (2.2), then we have:

$$\begin{aligned}
\sum_{m=1}^{\infty} f(m) z^m &= \sum_{m=1}^{\infty} \sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} h_{m-1}\left(\frac{q^i}{1-q^i}, \frac{q^{i+1}}{1-q^{i+1}}, \dots, \frac{q^n}{1-q^n} \right) z^m \\
&\overset{(1.4)}{=} z \sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \frac{1}{(1-q^i z/(1-q^i)) \dots (1-q^n z/(1-q^n))} \\
&= z \sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \sum_{l=0}^{\infty} (q^i;q)_{n-i+1} \begin{bmatrix} n-i+l \\ l \end{bmatrix} (q^i(1+z))^l \\
&= z \begin{bmatrix} n \\ k \end{bmatrix} \sum_{i=k}^{n} \frac{(q;q)_{n-k}}{(q;q)_{i-k}} \sum_{l=0}^{\infty} \begin{bmatrix} n-i+l \\ l \end{bmatrix} q^{(l+1)i} \sum_{m=0}^{l} \binom{l}{m} z^m \\
&= z \begin{bmatrix} n \\ k \end{bmatrix} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} z^m \sum_{i=k}^{n} \frac{(q;q)_{n-k}}{(q;q)_{i-k}} \begin{bmatrix} n-i+l \\ l \end{bmatrix} q^{(l+1)i} \\
&\overset{(2.4)}{=} \begin{bmatrix} n \\ k \end{bmatrix} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} z^{m+1} q^{(l+1)k}
\end{aligned}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^k z}{1-q^k(1+z)} = \sum_{m=1}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} \frac{q^{km}}{(1-q^k)^m} z^m \,.$$

Taking $\mathbb{X} = \{1\}$, $\mathbb{Y} = \{q, q^2, \ldots\}$ and $\mathbb{Z} = \{zq, zq^2, \ldots\}$, we get from (1.5) and (1.6), the proof of (2.3):

$$\begin{aligned} \frac{(zq;q)_n}{(q;q)_n} &= h_n((\mathbb{X}+\mathbb{Y})-\mathbb{Z}) \\ &= h_n(\mathbb{X}+(\mathbb{Y}-\mathbb{Z})) = \sum_{i=0}^{n} h_i(\mathbb{Y}-\mathbb{Z}) h_{n-i}(\mathbb{X}) = \sum_{i=0}^{n} \frac{(z;q)_i}{(q;q)_i} q^i. \end{aligned}$$

∎

Taking

$$f(x) \;=\; \sum_{i=1}^{n} \frac{(-1)^{i-1}(x^i-(-1)^i)}{1-q^i} q^i \sum_{i \le i_2 \le \ldots \le i_m \le n} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1-q^{i_j})},$$

and $\mathbb{X} = \{-1, -q, -q^2, \ldots\}$, then we have,

$$\begin{aligned} f(x) &= f(x_1) + \sum_{k=1}^{n} f(x_1)\partial_1 \ldots \partial_k \, (x+1)\ldots(x+q^{k-1}) \\ &= \sum_{k=1}^{n} \sum_{i=k}^{n} (-1)^{k-1} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \sum_{i \le i_2 \le \ldots \le i_m \le n} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1-q^{i_j})} (x+1)\ldots(x+q^{k-1}) \\ &= \sum_{k=1}^{n} (-1)^{k-1}(x+1)\ldots(x+q^{k-1}) \sum_{i=k}^{n} \begin{bmatrix} i \\ k \end{bmatrix} \frac{q^i}{1-q^i} \sum_{i \le i_2 \le \ldots \le i_m \le n} \frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m (1-q^{i_j})} \\ &= \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^{k-1}(x+1)\ldots(x+q^{k-1})}{(1-q^k)^m} q^{mk}, \end{aligned}$$

as stated in (1.1). ∎

Taking

$$f(x) = \sum_{i=0}^{n} (-1)^{i-1} \frac{(z;q)_i}{(q;q)_i} x^i q^i, \quad \text{and} \quad \mathbb{X} = \{-1, -q, -q^2, \ldots\},$$

we have,

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} f(x_1)\partial_1 \ldots \partial_k \, (x+1)\ldots(x+q^{k-1}) \\ &= \sum_{k=0}^{n} (x+1)\ldots(x+q^{k-1}) \sum_{i=k}^{n} (-1)^{k-1} \begin{bmatrix} i \\ k \end{bmatrix} \frac{(z;q)_i}{(q;q)_i} q^i \\ &= \sum_{k=0}^{n} (-1)^{k-1}(x+1)\ldots(x+q^{k-1}) q^k \frac{(z;q)_k}{(q;q)_k} \sum_{i=k}^{n} \frac{(zq^k;q)_{i-k}}{(q;q)_{i-k}} q^{i-k} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n}(-1)^{k-1}(x+1)\dots(x+q^{k-1})q^k\frac{(z;q)_k}{(q;q)_k}\frac{(zq^{k+1};q)_{n-k}}{(q;q)_{n-k}}\\
&= \frac{(z;q)_{n+1}}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^{k-1}(x+1)\dots(x+q^{k-1})}{1-zq^k}q^k,
\end{aligned}
$$

and this proves (1.2). ∎

3. Special Cases

In their study of overpartitions [3, Theorem 4.4], Corteel and Lovejoy obtained a combinatorial interpretation of the identity:

$$
\sum_{i=1}^{\infty}(-1)^{i-1}\frac{(-1;q)_i}{(q;q)_i}\frac{q^i}{1-q^i}=\sum_{i=1}^{\infty}\frac{2q^{2i-1}}{1-q^{2i-1}}=\sum_{i=1}^{\infty}\frac{2q^i}{1-q^{2i}}\,. \tag{3.1}
$$

Corteel asked us a proof of the following two related identities :

$$
\sum_{i=1}^{2n-1}\begin{bmatrix}2n-1\\i\end{bmatrix}\frac{(-1)^{i-1}(-1;q)_i}{(1-q^i)^m}q^{mi}
=\sum_{i=1}^{n}\frac{2q^{2i-1}}{1-q^{2i-1}}\sum_{2i-1\le i_2\le\dots\le i_m\le 2n-1}\frac{q^{\sum_{j=2}^m i_j}}{\prod_{j=2}^m(1-q^{i_j})}, \tag{3.2}
$$

and

$$
\sum_{i=1}^{2n}\begin{bmatrix}2n\\i\end{bmatrix}\frac{(-1)^{i-1}(-1;q)_i}{1-q^{i+2}}q^i=\sum_{i=1}^{n}\frac{2q^{2i-1}(1-q)}{(1+q^2)(1-q^{2i-1})(1-q^{2i+1})}\,, \tag{3.3}
$$

the first one becoming (3.1) for $m=1$ and $n=\infty$, because

$$
\sum_{i=1}^{\infty}\frac{2q^i}{1-q^{2i}}=\sum_{i=1}^{\infty}\frac{2q^{2i-1}}{1-q^{2i-1}}\,.
$$

In fact, (3.2) is the special case of (1.1) when $x=1$.

Changing n into $2n$, specializing $x=1$ and $z=q^2$, one gets from (1.2)

$$
\frac{(1-q^{2n+1})(1-q^{2n+2})}{1-q}\sum_{0}^{2n}\begin{bmatrix}2n\\i\end{bmatrix}\frac{(-1)^{i-1}(-1;q)_i}{1-q^{i+2}}q^i=\sum_{0}^{2n}(-1)^{i-1}q^i\frac{1-q^{i+1}}{1-q}\,.
$$

Suppressing in the left member the term corresponding to $i=0$, and taking into account that the right member of (3.3) sums to $2q(1-q^{2n})(1-q^4)^{-1}(1-q^{2n+1})^{-1}$, one obtains (3.3).

The case $x=0$, $m=1$ of (1.1) is due to Van Hamme [6] (see also [2], [5], [7]):

$$
\sum_{i=1}^{n}\begin{bmatrix}n\\i\end{bmatrix}\frac{(-1)^{i-1}q^{\binom{i+1}{2}}}{1-q^i}=\sum_{i=1}^{n}\frac{q^i}{1-q^i}.
$$

Taking $x = 0$, and (1.7), we get the formula of Dilcher [4]:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (-1)^{i-1} \frac{q^{\binom{i}{2}+mi}}{(1-q^i)^m} = \sum_{1 \le i_1 \le i_2 \le \ldots \le i_m \le n} \frac{q^{i_1}}{1-q^{i_1}} \cdots \frac{q^{i_m}}{1-q^{i_m}}.$$

When $x = 0$ and $z = q^m$ in (1.2), we get Uchimura's identity [8]:

$$\sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} \frac{(-1)^{i-1} q^{\binom{i+2}{2}}}{1-q^{i+m}} = \sum_{i=1}^{n} \frac{q^i}{1-q^i} \Big/ \begin{bmatrix} i+m \\ i \end{bmatrix}.$$

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